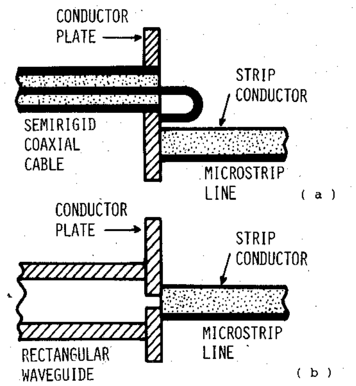

Fig. 2. Coupling structures between one end of the microstrip line resonator and (a) a coaxial cable (2–18 GHz), or (b) a waveguide (35 and 50 GHz).

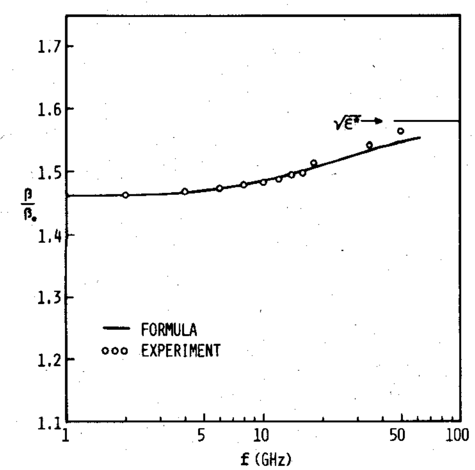

Fig. 3. The measured dispersion of a microstrip line with a Fluorglas substrate. $\epsilon^*=2.5$; $w/h=3.04$; $h=1.15$ mm.

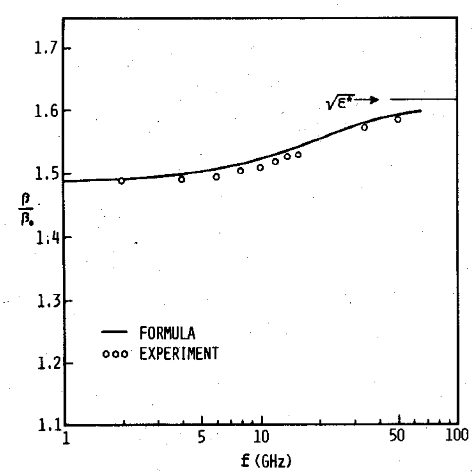

Fig. 4. The measured dispersion of a microstrip line with a Rexolite substrate. $\epsilon^*=2.62$; $w/h=2.82$; $h=1.57$ mm.

resonator is made of a microstrip line shorted at both ends by conductor plates. λ is measured as the resonator length divided by the number of standing waves between the conductor plates. λ_0 is the wave velocity divided by the frequency reading on the microwave counter (2–18 GHz) or wave meter (35 and 50 GHz). The ratio, λ_0/λ, equals to β/β_0. Fig. 2 illustrates two types of coupling structures between one end of the microwave resonator

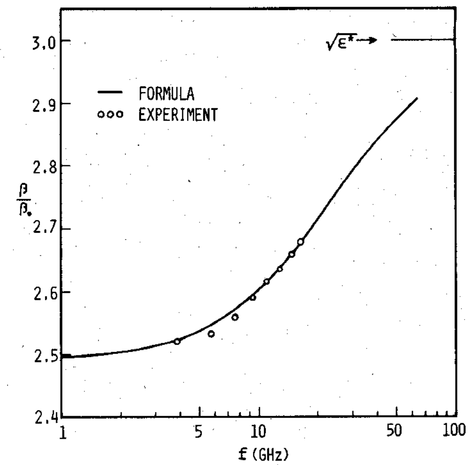

Fig. 5. The measured dispersion of a microstrip line with a Alumina substrate. $\epsilon^*=9.0$; $w/h=0.867$; $h=0.97$ mm.

and a coaxial cable (2–18 GHz) or a waveguide (35 and 50 GHz) in the experimental setup.

Figs. 3, 4, and 5 show the results of measurements by using the above experimental setup for three substrates: Fluorglas ($\epsilon^*=2.5$), Rexolite ($\epsilon^*=2.62$), and Alumina ($\epsilon^*=9.0$). Reasonable agreement between the formula and experimental results in these figures indicates the practicality of the approximate formula.

Acknowledgment

The authors thank Dr. Y. Suzuki for his helpful comments.

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Characteristic Impedances of Four-Conductor Transmission Line

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***Abstract*— A general formula for calculation of the characteristic impedance of four conductor transmission line in a rectangular shield is derived. A number of coupled and single strip transmission lines are considered by simplifying the general formula. Numerical results for a line in a square shield are presented graphically.**

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I. Introduction

The purpose of this correspondence is to present a general formula for characteristic impedances of a transmission line consisting of four rectangular bars symmetrically situated in a metal shield. The results presented below are obtained by using the self-consistent field method (SCFM) [1], [2]. This method is based on the assumption that the investigated transmission line supports TEM waves and the electromagnetic problem reduces to solving the Poisson equation $\Delta A = -\mu J$, for the z-component of the vector potential $A = z_0 A_z(x, y)$. The unknown current density $J = z_0 J_z(x, y)$ on the conductors surfaces is determined by the SCFM in accordance with the boundary condition $n \times H_t = J$, where n is a unit vector normal to the k-surface of the bar and H_t is the tangential magnetic field near this surface, defined by relation $H = \frac{1}{\mu}\nabla \times A$. The characteristic impedance of the line is calculated from the expression $Z = L/\sqrt{\epsilon\mu}$, where $L = 1/I^2 \int A \cdot J\, ds$ is the inductance per unit length of the investigated line, $\epsilon = \epsilon_0\epsilon_r$ and $\mu = \mu_0\mu_r$ are constants of the medium and I is the current on the one of line's conductors.

II. Theory

The investigated four conductor transmission line is shown in Fig. 1. As it is known [3], there are four different modes of TEM excitation between the conductors of the line—see Fig. 2. When the plane $x=0$ (or $y=0$), is equivalent to an electric wall the so-called odd-mode of excitation is obtained, and even-mode, when the corresponding plane is equal to a magnetic wall. The planes $x=0$; $y=0$ bisect the cross section of the line into four images "cells." So, it follows that the characteristic impedance of any mode of excitation can be determined for a single cell. It is convenient to denote the different modes of excitation with two indexes—e (even) and o (odd). The first index characterizes the excitation of the horizontal conductors (with respect to $x=0$), and the second—of the vertical conductors (with respect to $y=0$). For example, Z^{eo} denotes the characteristic impedance of the four conductor line with even–odd mode of excitation, as shown in Fig. 2(a).

Taking into consideration the boundary conditions on the plane $x=0, y=0$ and on the metal shield, a general expression for the z-component of the potential can be written in the form

$$A_z(x,y) = \mu\frac{4}{hb} \cdot \sum_{m,n=1}^{\infty} \frac{\begin{bmatrix} \sin n\pi x/b \\ \cos(m-1/2)\pi x/b \end{bmatrix}^{o}_{e} \begin{bmatrix} \sin m\pi y/h \\ \cos(m-1/2)\pi y/h \end{bmatrix}^{o}_{e}}{\left[\frac{\pi}{b}\begin{pmatrix} n \\ n-1/2 \end{pmatrix}^{o}_{e}\right]^2 + \left[\frac{\pi}{h}\begin{pmatrix} m \\ m-1/2 \end{pmatrix}^{o}_{e}\right]^2} \cdot \left\{ \int_0^h \int_0^b J_z(x',y') \cdot \begin{bmatrix} \sin n\pi x'/b \\ \cos(n-1/2)\pi x'/b \end{bmatrix}^{o}_{e} \begin{bmatrix} \sin m\pi y'/h \\ \cos(m-1/2)\pi y'/h \end{bmatrix}^{o}_{e} dx'\,dy' \right\} \tag{1}$$

where the current density is defined by

$$J_z(x,y) = \begin{cases} \left.\begin{matrix} J_1(x)\,\delta(y-d) \\ J_2(x)\,\delta(y-d-t) \end{matrix}\right\}, & s \leqslant x \leqslant s+w \\ \left.\begin{matrix} J_3(y)\,\delta(x-s) \\ J_4(y)\,\delta(x-s-w) \end{matrix}\right\}, & d \leqslant y \leqslant d+t. \end{cases} \tag{2}$$

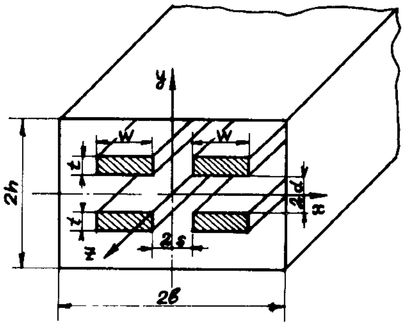

Fig. 1 Four-conductor transmission line in a rectangular shield.

In accordance with the SCFM procedure the functions $J_k(1)$ are substituted by their average values $\tilde{J}_k = 1/l_k \int J_k(l)\, dl$. Further, performing the sequence of calculations: $(\tilde{J}_k = 1) \to A_z(x,y) \to H_t(x,y) \to J_k(1) \to \tilde{J}_k \to \cdots$, the final result for the average densities is

$$\tilde{J}_{k,p} = \frac{1}{\pi^2}\sum_{l=1}^{4} \tilde{J}_{l,p-1} \sum_{m=1}^{\infty} \frac{\alpha_{km}\beta_{lm}\gamma_{lkm}}{\binom{m}{m-1/2}^2\left[1 \mp e^{-\binom{m}{m-1/2}2\pi B}\right]} \tag{3}$$

where p is number of iteration, $\tilde{J}_{l,1} = 1$, and

$$\alpha_{1m} = \frac{1}{W}\begin{bmatrix} \cos m\pi D \\ -\sin(m-1/2)\pi D \end{bmatrix}$$

$$\alpha_{2m} = \frac{1}{W}\begin{bmatrix} -\cos m\pi(D+T) \\ \sin(m-1/2)\pi(D+T) \end{bmatrix}$$

$$\alpha_{3m} = \alpha_{4m} = \frac{1}{T}\begin{bmatrix} \cos m\pi D - \cos m\pi(D+T) \\ \sin(m-1/2)\pi(D+T) - \sin(m-1/2)\pi D \end{bmatrix}$$

$$\beta_{1m} = \begin{bmatrix} \sin m\pi D \\ \cos(m-1/2)\pi D \end{bmatrix}$$

$$\beta_{2m} = \begin{bmatrix} \sin m\pi(D+T) \\ \cos(m-1/2)\pi(D+T) \end{bmatrix}$$

$$\beta_{3m} = \beta_{4m} = \begin{bmatrix} \cos m\pi D - \cos m\pi(D+T) \\ \sin(m-1/2)\pi(D+T) - \sin(m-1/2)\pi D \end{bmatrix}$$

$$\gamma_{11m} = \gamma_{12m} = \gamma_{21m} = \gamma_{22m} = \binom{m}{m-1/2}2\pi W\left(1 \mp e^{-\binom{m}{m-1/2}2\pi B}\right) - 2\left[1 - e^{-\binom{m}{m-1/2}\pi W}\right] \cdot \left[1 \mp e^{-\binom{m}{m-1/2}(2B-W)\pi}\right] \mp \left[1 - e^{-\binom{m}{m-1/2}\pi W}\right]^2 \cdot \left[e^{-\binom{m}{m-1/2}2\pi S} \pm e^{-\binom{m}{m-1/2}2\pi(B-S-W)}\right]$$

$$\gamma_{13m} = \gamma_{23m} = \left[1 - e^{-\binom{m}{m-1/2}\pi W}\right] \cdot \left[1 \mp e^{-\binom{m}{m-1/2}2\pi S}\right]\left[1 - e^{-\binom{m}{m-1/2}\pi(2B-2S-W)}\right]$$

$$\gamma_{14m} = \gamma_{24m} = \left[1 - e^{-\binom{m}{m-1/2}\pi W}\right] \cdot \left[1 \mp e^{-\binom{m}{m-1/2}\pi(2S+W)}\right]\left[1 - e^{-\binom{m}{m-1/2}2\pi(B-S-W)}\right]$$

$$\gamma_{31m} = \gamma_{32m} = \left[1 - e^{-\binom{m}{m-1/2}\pi W}\right] \cdot \left[1 \pm e^{-\binom{m}{m-1/2}2\pi S}\right]\left[1 - e^{-\binom{m}{m-1/2}\pi(2B-2S-W)}\right]$$

$$\gamma_{41m} = \gamma_{42m} = \left[1 - e^{-\binom{m}{m-1/2}\pi W}\right] \cdot \left[1 \mp e^{-\binom{m}{m-1/2}\pi(2S+W)}\right]\left[1 + e^{-\binom{m}{m-1/2}2\pi(B-S-W)}\right]$$

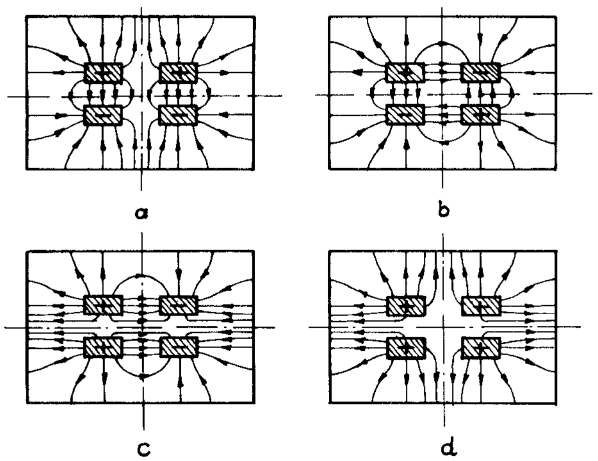

Fig. 2. Excitation of the conductors. (a) Even-odd mode. (b) Odd-odd mode. (c) Odd-even mode. (d) Even-even mode.

$$\gamma_{33m}=\left[1\pm e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi S}\right]\left[1-e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi(B-S)}\right]$$

$$\gamma_{34m}=e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)\pi W}\left[1\pm e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi S}\right]\left[1-e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi(B-S-W)}\right]$$

$$\gamma_{43m}=e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)\pi W}\left[1\mp e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi S}\right]\left[1+e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi(B-S-W)}\right]$$

$$\gamma_{44m}=\left[1\mp e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi(S+W)}\right]\left[1+e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi(B-S-W)}\right].$$

In the above expressions the dimensions of the line are normalized with respect to the height of the shield h as follows: $B=b/h$, $S=s/h$, $D=d/h$, $W=w/h$, and $T=t/h$. For simplicity, the indexes e and o are omitted. The combination m and upper sign corresponds to odd-odd mode of excitation, m and lower sign to even-odd mode, $m-1/2$ and upper sign to odd-even mode and $m-1/2$ and lower sign to even-even mode.

The substitution of the potential (1) into the integral for the inductance L gives then the following expression for the characteristic impedance calculation:

$$Z=\sqrt{\frac{\mu}{\epsilon}}\sum_{m,n=1}^{\infty}\frac{4/hb}{\left[\frac{\pi}{b}\left(\begin{matrix}n\\ n-1/2\end{matrix}\right)_e^o\right]^2+\left[\frac{\pi}{h}\left(\begin{matrix}m\\ m-1/2\end{matrix}\right)_e^o\right]^2}\left[\frac{\int_0^h\int_0^b J_z(x,y)\left[\begin{matrix}\sin n\pi x/b\\ \cos(n-1/2)\pi x/b\end{matrix}\right]_e^o\left[\begin{matrix}\sin m\pi y/h\\ \cos(m-1/2)\pi y/h\end{matrix}\right]_e^o dx\,dy}{\int_0^h\int_0^b J_z(x,y)\,dx\,dy}\right]^2. \quad (4)$$

Equation (4) is a stationary function with respect to the current distribution $J_z(x,y)$. Therefore, the substitution $J_k(1)\to\tilde{J}_k$ will lead to a small error for Z—less than 2–3 percent.

Finally, the general design formula for the characteristic impedances of the investigated four-conductor line is

$$Z\sqrt{\frac{\epsilon_r}{\mu_r}}=\frac{120}{\pi^2}\sum_{i,j=1}^{4}\frac{\tilde{J}_i\tilde{J}_j}{\left[W(\tilde{J}_1+\tilde{J}_2)+T(\tilde{J}_3+\tilde{J}_4)\right]^2}\cdot\sum_{m=1}^{\infty}\frac{\beta_{im}\beta_{jm}Z_{ijm}}{\left(\begin{matrix}m\\ m-1/2\end{matrix}\right)^3\left[1\mp e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi B}\right]} \quad (5)$$

where the matrix elements $Z_{ijm}=Z_{jim}$ are expressed as follows:

$$Z_{11m}=Z_{12m}=Z_{22m}=\gamma_{11m}$$

$$Z_{13m}=Z_{23m}=\gamma_{13m};\quad Z_{14m}=Z_{24m}=\gamma_{14m}$$

$$Z_{33m}=\left[1\mp e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi S}\right]\left[1-e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi(B-S)}\right]$$

$$Z_{34m}=e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)\pi W}\left[1\mp e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi S}\right]\left[1-e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi(B-S-W)}\right]$$

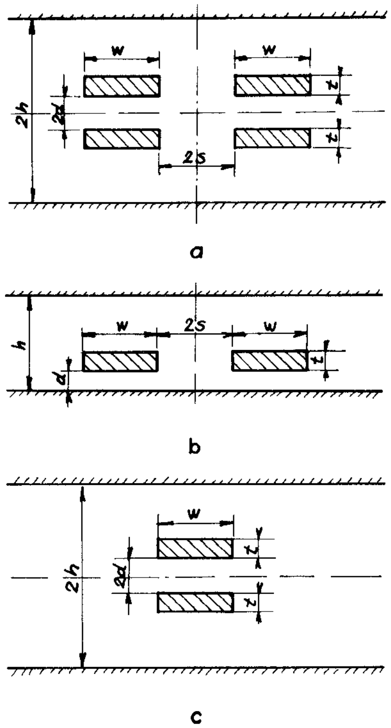

Fig. 3. Transmission lines between parallel plates. (a) Four conductor line. (b) Narrow-side coupled strip transmission line. (c) Broad-side coupled strip transmission line.

$$Z_{44m}=\left[1\mp e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi(S+W)}\right]\left[1-e^{-\left(\begin{smallmatrix}m\\ m-1/2\end{smallmatrix}\right)2\pi(B-S-W)}\right].$$

III. Special Cases

The obtain formulas (3) and (5) can be used also for the case when the side walls of the shield are displaced to infinity ($B\to\infty$)—see Fig. 3(a). For this case in (3) and (5) the exponents containing the factor B are equal to zero. If the substitution $B\to\infty$ is taken into consideration, the narrow-side coupled strip transmission line shown in Fig. 3(b) can be characterized by the impedances Z^{eo} and Z^{oo} [4]. The broad-side coupled strip transmission line shown in Fig. 3(c) is characterized by the impedances Z^{oe} and Z^{oo} in the case when both B and S tend to infinity [5].

The expression for the characteristic impedance of the odd-odd mode of excitation Z^{oo} corresponds to the rectangular coaxial line (see Fig. 4(a), investigated earlier [1]. A result for the impedance of the trough line—Fig. 4(b), can be obtained if in the expression for Z^{oo} the factor B is set to infinity. When both dimensions B and S tend to infinity, the formula for Z^{oo} simplifies to the expression for the characteristic impedance of the unbalanced strip transmission line [6].

IV. Numerical Results

The calculation of the characteristic impedances is made by computer. The requirement $|Z_p-Z_{p+1}|/Z_p<10^{-3}$ serves as a criterion for determination the number of iterations p in (3). The numerical results for the characteristic impedances Z^{eo}, Z^{oo}, Z^{oe}

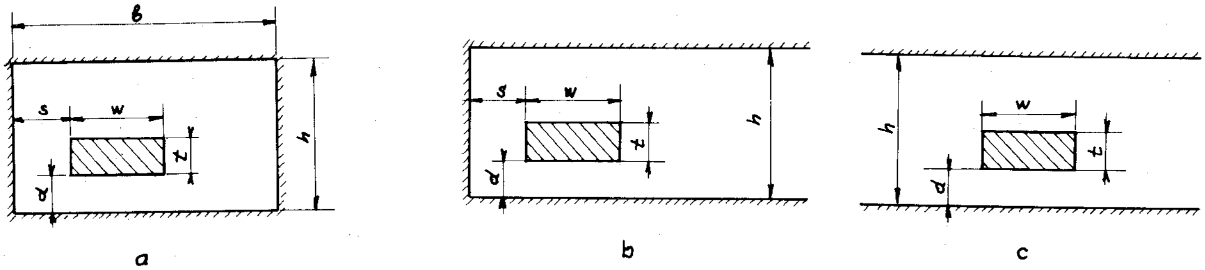

Fig. 4. Single transmission lines. (a) Rectangular coaxial line. (b) Trough line. (c) Unbalanced strip line.

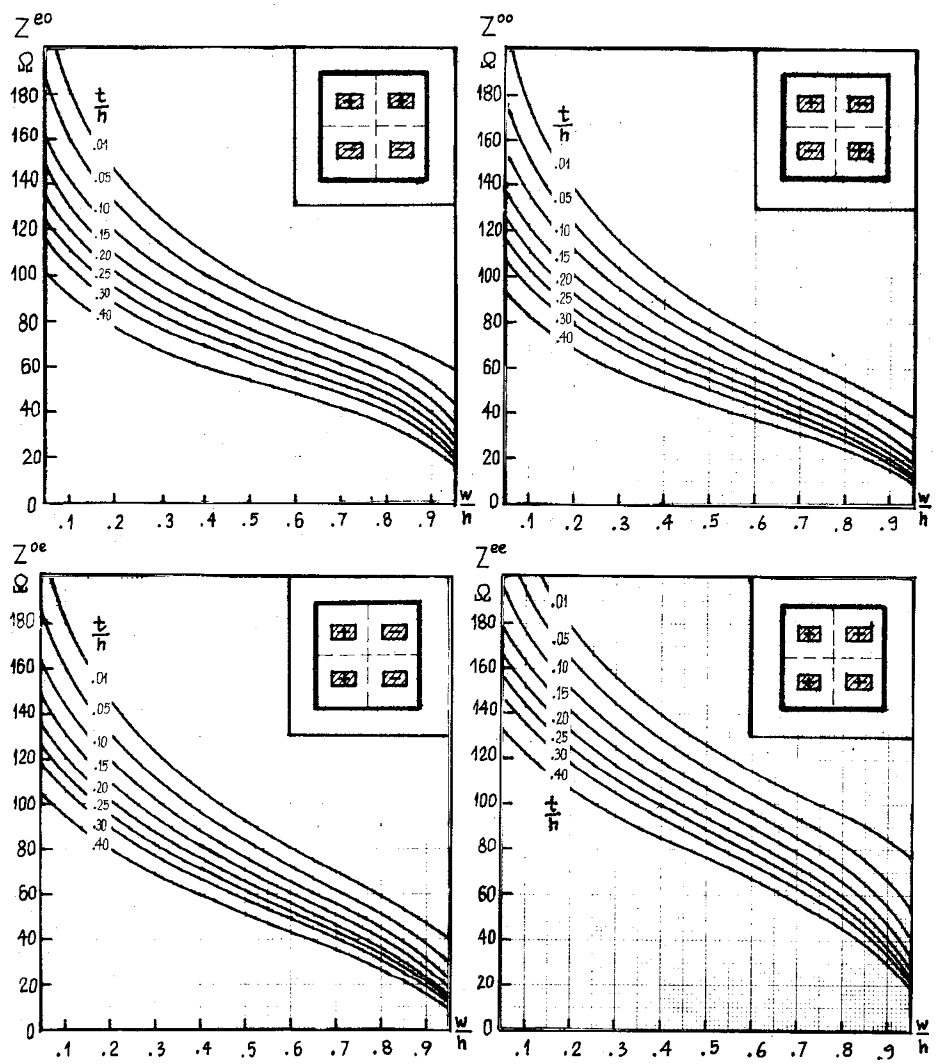

Fig. 5. Characteristic impedances of four conductor line in a square shield vers w/h and t/h at $\epsilon_r = \mu_r = 1$, $2s + w = b$ and $2d + t = h$.

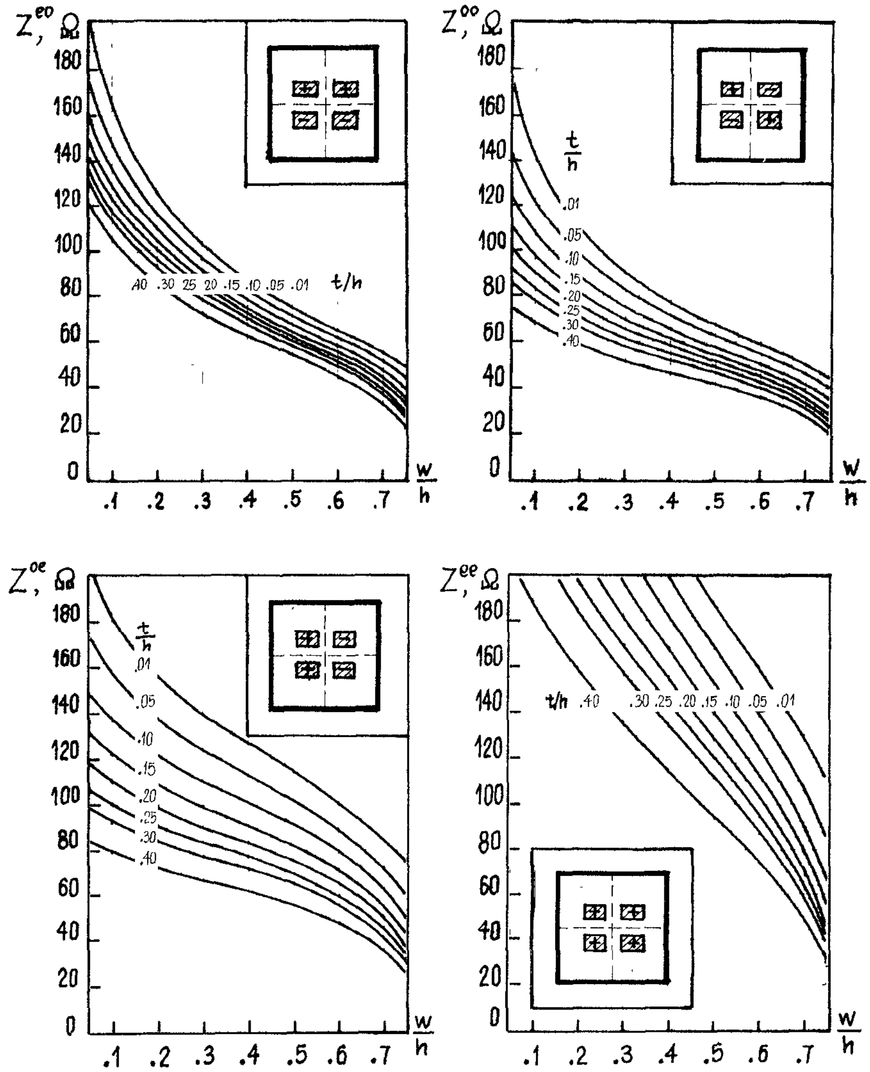

Fig. 6. Characteristic impedances of four-conductor line in a square shield vers w/h and t/h at $\epsilon_r=\mu_r=1$, $s/h=0.2$ and $d/h=0.2$.

and Z^{ee} of the four-conductor line with a square shield ($B=1$), are presented graphically in Fig. 5—for the case $2s+w=b$, $2d+t=h$ and in Fig. 6—when $s/h=0.2$, $d/h=0.2$. The comparison of data for the value of the impedances Z^{eo} and Z^{oo} from Fig. 5 with the corresponding results calculated from Getsinger's graphs [7] gives agreement within 2–3 percent.

The numerical data for the pair of the characteristic impedances Z^{eo}, Z^{oo} and Z^{oe}, Z^{oo} can be used for the design of coupled transmission lines in square shield [8]. The graphs for the other lines shown in Figs. 3 and 4 are presented in papers [4]–[6].

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On the Orthogonality of Approximate Waveguide Mode Functions

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***Abstract*—For many waveguides, only approximate solutions for the mode functions are available and in such cases the question arises, whether the orthogonality property of the exact modes can be preserved. This problem is addressed in the present paper. A fairly general method of**

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